V Tensor Decomposition

Given $T = \sum_{i=1}^{r} \left(u^{(i)}\right)^{\otimes 3}$ $\quad u^{(i)} \in \mathbb{R}^n$

Goal: recover $\{u^{(1)}, \ldots, u^{(r)}\}$

Random Components

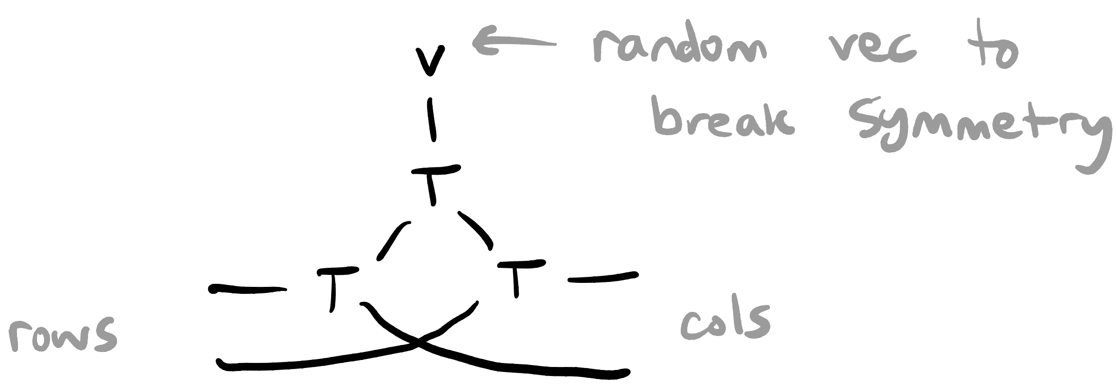
$u^{(i)} \sim N(0, I_n)$

Known poly-time approaches succeed for

$$r \ll n^{3/2}$$

hides polylog(n)

[Hopkins, Schramm, Shi, Steurer '15] $r \ll n^{4/3}$



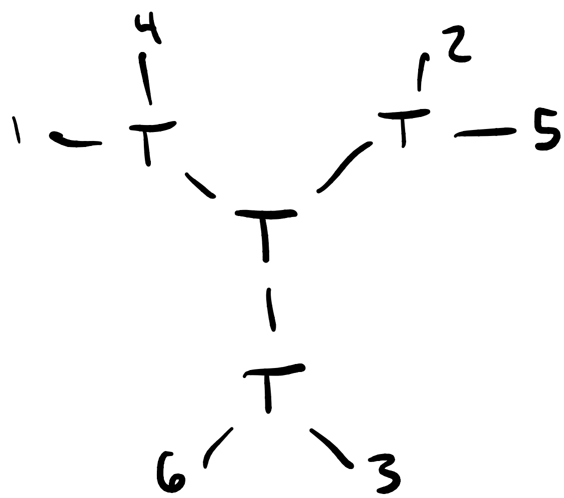
Hope: recover some $u^{(i)}$

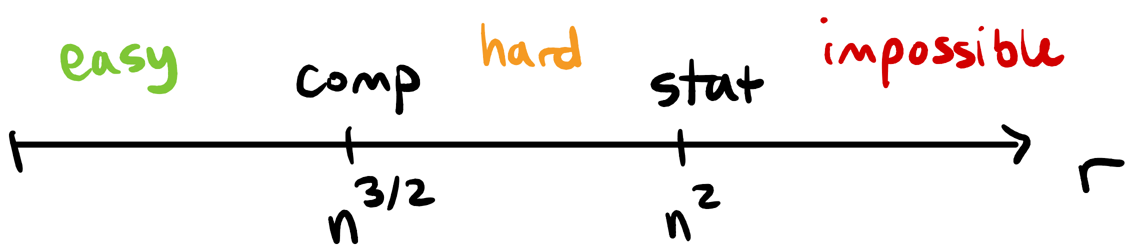
Repeat with different v

[Ma, Shi, Steurer '16] $r \ll n^{3/2}$ using sum-of-squares

[Ding, d'Orsi, Liu, Steurer, Tiegel '22] $r \ll n^{3/2}$

↳ Based on $\{1,2,3\}, \{4,5,6\}$ flattening of





Identifiable when $r \lesssim n^2$ [Bocci, Chiantini, Ottaviani '14]

↳ Related: max rank of $n \times n \times n$ is $\leq n^2$


n slices, each rank $\leq n$

Hard for low-deg polynomial when $r \gg n^{3/2}$ [W '23]

Generic Components

Random components is a strong assumption

Can't hope for arbitrary components (NP-hard)

Sweet spot: "generic" components

↳ Succeed for "almost all" inputs

<u>Def</u> Predicate $P(x)$ holds for "generically chosen" $x \in \mathbb{R}^n$ if $\exists$ polynomial $f: \mathbb{R}^n \to \mathbb{R}$, not identically zero, such that

$$f(x) \neq 0 \quad \Rightarrow \quad P(x) \text{ holds}$$

* "Bad" values for x: zero set of some polynomial $\Rightarrow$ measure zero

Setting: $$T = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$

where $u^{(i)}, v^{(i)}, w^{(i)}$ are generically chosen from $\mathbb{R}^{3rn}$

Goal: recover the collection of rank-1 terms

$$u \otimes v \otimes w = (2u) \otimes \left(\tfrac{1}{2} v\right) \otimes w$$

Focus: $p = 3, 4$ (tensor order)

Easier task: "rank detection"

↳ find r

- Prerequisite for decomp
- Turns out to have same threshold as decomp...
- Identifiable when $r \lesssim n^{p-1}$ (same as random)

Approach: flattening $\quad T \mapsto M(T)$

$$\mathrm{rank}(T) \underset{?}{\rightsquigarrow} \mathrm{rank}(M)$$

<u>$p=4$</u> $\quad T = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)} \otimes x^{(i)}$

$$T \mapsto M = M(T) \qquad = T = \quad \begin{array}{cc} & n^2 \\ n^2 & \square \end{array}$$

$$u \otimes v \otimes w \otimes x \mapsto \underbrace{(u \otimes v)}_{n^2} \underbrace{(w \otimes x)^T}_{n^2}$$

rank-1 $\longmapsto$ rank-1

rank-r $\longmapsto$ rank-?

r, provided

- components generic
- $r \le n^2$

$\hookrightarrow$ $n^2 \times n^2$ matrix has max rank n^2

$\Rightarrow$ rank detection for $r \le n^2$

Also: decomp for $r \le cn^2$

$\hookrightarrow$ $\mathrm{colspan}(M) = \mathrm{span}\{u^{(i)} \otimes v^{(i)} : i \in [r]\}$

Search for rank-1 vectors in subspace

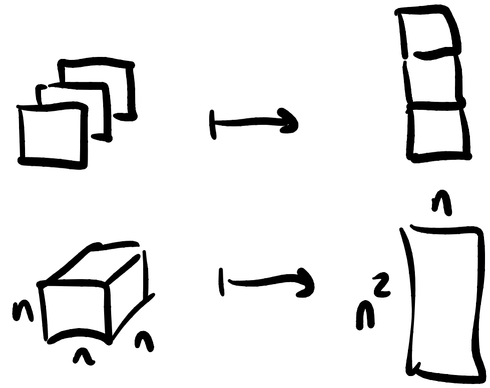
[De Lathauwer, Castaing, Cardoso '07] "FOOBI"

[Johnston, Lovitz, Vijayaraghavan '22]

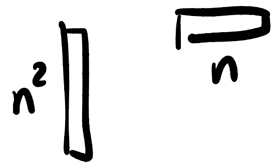
* For $p=4$, generic is no harder than random (both $r \lesssim n^2$)

p = 3 $\quad T = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$

"Trivial" flattening



$$u \otimes v \otimes w \longmapsto (u \otimes v)\, w^T$$



rank-1 $\longmapsto$ rank-1

rank-r $\longmapsto$ rank-?

r, provided

- components generic
- $r \le n$

$\Rightarrow$ rank det for $r \le n$

Also: decomp for $r \le n$

"simultaneous diagonalization"

"Jennrich's algorithm"

(classical ~1970)

$r \le n$ "undercomplete" $\leftarrow$ known algs

$r > n$ "overcomplete" $\leftarrow$?

* Recall: for random components, $r \lesssim n^{3/2}$

* Generic-random gap for odd-order ?

Some results for overcomplete case

[Persu '18] $r \leq \frac{3}{2} n$ (rank det only)

[Chen, Rademacher '20] $r = n + k$ w/ runtime $n^{O(k)}$

[Koiran '24] $r \leq \frac{4}{3} n$

[Kothari, Moitra, W '24+] $r \leq (2-\varepsilon) n$

↳ Based on Koszul-Young flattening

[Landsberg, Ottaviani '13]

Param $p \geq 1$ (integer, large const)

Linear map $T \mapsto M(T)$

$$u \otimes v \otimes w \longmapsto A(u) \otimes (v w^T)$$

Kronecker product

$$\binom{2p+1}{p} \times \binom{2p+1}{p+1}: \quad \boxed{A(u)}$$

* square matrix

Ex $(p=1)$

$$A(u) = \begin{array}{c} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{array}{c} \begin{array}{ccc} \{1,2\} & \{1,3\} & \{2,3\} \end{array} \\ \begin{bmatrix} -u_2 & -u_3 & 0 \\ u_1 & 0 & -u_3 \\ 0 & u_1 & u_2 \end{bmatrix} \end{array}$$

$A(u)_{su} \neq 0$ when $S \subseteq U$

$\operatorname{rank}(M) = \binom{2p}{p} \cdot \operatorname{rank}(T)$

provided

- components generic
- $r \leq (2-\varepsilon)n$

$\varepsilon = \varepsilon(p)$

$p \to \infty \Rightarrow \varepsilon \to 0$