IV Kikuchi Matrices [W, Alaoui, Moore '19]

Motivating question: How hard is the hard regime?

Even-order tensor PCA: $T = \lambda v^{\otimes p} + W$

p even

assume $v \in \{\pm 1\}^n$

Recall: "possible but hard" regime $(p > 2)$

$$n^{(1-p)/2} \ll \lambda \ll n^{-p/4}$$

<u>Fact</u>: In hard regime, there are algorithms of runtime $n^{O(\ell)}$ $(1 \le \ell \le n)$ when

$$\lambda \gg \ell^{1/2 - p/4} n^{-p/4}$$

hides polylog(n) factor

[Bhattiprolu, Guruswami, Lee '16]

* Interpolates smoothly between poly-time $\ell = O(1)$ and brute force search $\ell \approx n$

Conj: For $p \geq 3$, poly-time algs exist when

$\lambda \geq \varepsilon \cdot n^{-p/4}$ $\forall \varepsilon > 0$. runtime $n^{f(\varepsilon)}$

known for $\exists$

* "Smooth" computational phase transition, as opposed to "sharp" threshold

BBP, Kesten-Stigum, ...

Jump from poly time to fully exponential time $\exp(n^{1-o(1)})$

Kikuchi matrices (from "Kikuchi hierarchy")

Simple alg/proof achieving above tradeoff

* We believe this tradeoff is optimal (low-degree)

A particular flattening $T \longmapsto M(T)$

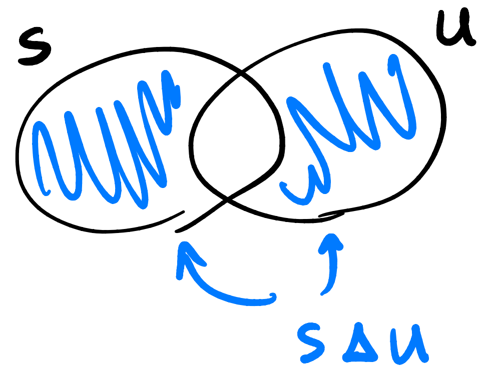
$\underbrace{n \times n \times \cdots \times n}_{p \text{ times}}$ (p even)

$M = (M_{SU}) \qquad S, U \subseteq [n] \qquad |S| = |U| = \ell$

$\uparrow$ Kikuchi level $\ell \geq \frac{p}{2}$

subset: $T_{\{1,2,3,4\}} = T_{1,2,3,4}$

$$M_{SU} = \begin{cases} T_{S \Delta U} & \text{if } |S \Delta U| = p \\ 0 & \text{otherwise} \end{cases}$$



<u>Ex</u> $\ell = p/2$:

$$M_{SU} = \begin{cases} T_{S \cup U} & \text{if } S, U \text{ disjoint} \\ 0 & \text{o.w.} \end{cases}$$

* Essentially the trivial/naive flattening/unfolding

* For larger ℓ, becomes sparse

Alg: $v_{max}(M) \in \mathbb{R}^{\binom{n}{\ell}}$

Entry S: estimate for $\prod_{i \in S} v_i \in \{\pm 1\}$

* Simple rounding procedure $\longrightarrow \hat{v} \in \mathbb{R}^n$

2 questions:

(i) How to analyze?

(ii) "Why" does it work?

(i) $T = X + W$

$$X = \lambda v^{\otimes p}$$

$$M(T) = M(X) + M(W)$$

$$M(W)_{SU} = \begin{cases} W_{S\Delta U} & \cdots \\ 0 & \cdots \end{cases}$$

Want $\|M(W)\| \leq \ldots$

$$\hookrightarrow M(W) = \sum_{\substack{V \subseteq [n] \\ |V| = p}} W_V \cdot M^{(V)}$$

W_V: Scalar iid $N(0,1)$

$M^{(V)}$: deterministic Matrix

$\longrightarrow$ Use standard matrix Chernoff bound

(ii) Why?

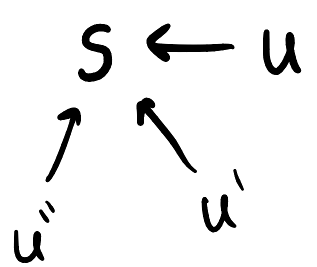
Leading eigenvec of M via power iteration:

$$m \leftarrow Mm \qquad m \in \mathbb{R}^{\binom{n}{\ell}}$$

$$m_S \leftarrow \sum_{U : |S \Delta U| = p} T_{S \Delta U}\, m_U$$

Message-passing scheme that uses ℓ-wise info

m_S : guess for $\prod_{i \in S} v_i \in \{\pm 1\}$



has m_U : guess for $\prod_{i \in U} v_i$

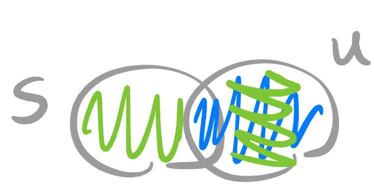
know $T_{S \Delta U}$: noisy estimate for $\underbrace{\prod_{i \in S \Delta U} v_i}_{(v^{\otimes p})_{S \Delta T}}$

$m_U \approx \prod_{i \in U} v_i$

$T_{S \Delta U} \approx \prod_{i \in S \Delta T} v_i$

$$T_{S \Delta U}\, m_U \approx \left(\prod_{i \in U} v_i\right)\left(\prod_{i \in S \Delta U} v_i\right) = \prod_{i \in S} v_i$$

$v_i \in \{\pm 1\} \qquad v_i^2 = 1$



* $T_{S \Delta U}\, m_U$ is "U's opinion about $\underbrace{\text{S's parity}}_{\prod_{i \in S} v_i}$"

* Average over many incoming "messages" (from each U)

* $|S \Delta U| = p$ is the only case where U has an "opinion" about S (given T)

Other uses for Kikuchi ...

- CSP refutation
- extremal combinatorics (hypergraph Moore bound)
- error-correcting codes

[Guruswami, Kothari, Manohar '21]
[Hsieh, Kothari, Mohanty '23]
[Alrabiah, Guruswami, Kothari, Manohar '23]